ELSEVIER

Journal of Geometry and Physics 43 (2002) 33–44

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Essential spectrum of the Lichnerowicz Laplacian on two tensors on asymptotically hyperbolic manifolds ☆

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Received 20 September 2001; received in revised form 4 December 2001

Abstract

On an n-dimensional asymptotically hyperbolic manifold with $n > 2$, we show that the essential spectrum of the Lichnerowicz Laplacian acting on trace free symmetric covariant two tensors is the ray $[(n-1)(n-9)/4, +\infty[$. For the particular case of the hyperbolic space, this is the spectrum.

MSC: 35P15; 58J50; 47A53

Subj. Class.: Differential geometry

Keywords: Asymptotically hyperbolic manifold; Lichnerowicz Laplacian; Symmetric two tensors; Essential spectrum; Asymptotic behavior

1. Introduction

The study of Laplacians acting on symmetric two tensor like the Lichnerowicz Laplacian Δ_L is very important to the understanding of some Riemannian geometric problems [3] and in general relativity. One of the problems is to find a metric with prescribed Ricci curvature [6], and the infinitesimal version of that problem is to invert the Lichnerowicz Laplacian on symmetric two tensor. In [5], I showed that the Ricci curvature can be arbitrarily prescribed in the neighborhood of the hyperbolic metric on the real hyperbolic space when the dimension is strictly larger than 9. The result given here shows in particular that 0 is in the spectrum for lower dimension, hence there certainly exist some obstructions to solve the Ricci equation.

☆ Research partially supported by the EU TMR project Stochastic Analysis and its Applications, ERB-FMRX-CT96-0075, and partially supported by the "ACI" program of the French Ministry of Research.
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0393-0440/02/$ – see front matter
PII: S0393-0440(02)00007-4

The action of the Lichnerowicz Laplacian in the conformal direction corresponds to the action of the Laplacian on function and we know that spectrum [13], this is why we are only interested here about the trace free direction.

The main result of this article is the following theorem.

Theorem. *On an n-dimensional asymptotically hyperbolic manifold with $n > 2$, the essential spectrum of the Lichnerowicz Laplacian acting on trace free symmetric covariant two tensors is the ray*

$$[\tfrac{1}{4}(n-1)(n-9), +\infty[.$$

For the hyperbolic space, this is the spectrum.

This theorem has also been proved by Lee [11, Proposition D] but I think that the short self-contained proof given here deserve to be in the literature.

We remark the fact that 0 is in the essential spectrum when $n \leq 9$.

For the hyperbolic space, the spectrum of the Lichnerowicz Laplacian on symmetric two tensor is the essential spectrum but in general, there might certainly exist some eigenvalues below $(n-1)(n-9)/4$, as is the case for functions [9].

For related results on the sphere or on real projective spaces, see [4].

We recall that the essential spectrum of Δ_{L} is the closed set

$$\sigma_{\mathrm{e}}(\Delta_{\mathrm{L}}) = \{\lambda \in \mathbb{R},\ \Delta_{\mathrm{L}} - \lambda Id : H^2 \to L^2 \quad \text{is not semi-Fredholm}\},$$

where *semi-Fredholm* means by definition here that the kernel is finite dimensional and the range is closed (there is no ambiguity because the operator is L^2 self-adjoint). So we have to look about the semi-Fredholm properties of the operators $\Delta_{\mathrm{L}} - \lambda Id$.

This article is organized as follows: we first recall a general criterion for an elliptic operator to be semi-Fredholm (Proposition 3.2). The problem is then reduced to find for all $\lambda < (n-1)(n-9)/4$, an asymptotic estimate

$$|(\Delta_{\mathrm{L}} - \lambda Id)u|_{L^2} \geq_{\infty} c|u|_{L^2},$$

near the boundary at infinity, where c is a positive constant and to show that this kind of estimate cannot exist when $\lambda \geq (n-1)(n-9)/4$. For the hyperbolic space, those estimates are global.

As we will see (Section 4), in order to obtain sharp estimates with the method used here, it is convenient to introduce an auxilliary Laplacian Δ_0 acting on symmetric two tensors and to use a Weitzenböck formula.

2. Definitions, notations and conventions

We describe here all objects we need throughout this paper.

Let $(\bar{M}, g)$ be a smooth, compact n-dimensional manifold with boundary. We assume that the boundary is the union of two closed submanifolds denoted $\partial_0 M$, the *inner boundary* and $\partial_\infty M$, the *boundary at infinity*, the last one not empty. Let $M := \bar{M} \backslash \partial_\infty M$ which is a non-compact manifold with boundary (when $\partial_0 M \neq \emptyset$). Let g be a Riemannian metric on M, we say that (M, g) is *conformally compact* if there exists a smooth defining function ρ on $\bar{M}$ (that is $\rho \in C^\infty(\bar{M})$, $\rho > 0$ on M, $\rho = 0$ on $\partial_\infty M$ and $\mathrm{d}\rho$ nowhere vanishing on $\partial_\infty M$) such that $\bar{g} = \rho^2 g$ is a smooth Riemannian metric on $\bar{M}$. Now if $|\mathrm{d}\rho|_{\bar{g}} = 1$ on $\partial_\infty M$, it is well known that g has asymptotically sectional curvature -1 (see [12] for example) near its boundary at infinity, in that case we say that (M, g) is *asymptotically hyperbolic*.

For any metric g on M, we denote ∇ as the associated Levi-Civita connexion, $\mathrm{Sect}(g)$ and $\mathrm{Ric}(g)$, respectively, the Sectional and the Ricci curvature of g.

We denote by $\mathcal{T}_p^q$ the set of rank p covariant and rank q contravariant tensors. When $p = 2$ and $q = 0$, we denote by $\mathcal{S}_2$ the subset of symmetric tensor which splits in $\mathcal{G} \oplus \mathcal{S}_{20}$ where $\mathcal{G}$ is the set of g-conformal tensors and $\mathcal{S}_{20}$ the set of trace free tensor (relative to g). We observe the summation convention, and we use g_{ij} and its inverse g^{ij} to lower or raise indices, with one exception: $\bar{g}^{ij}$ denotes the inverse of $\bar{g}_{ij}$, not the raised index version.

The Laplacian is defined as

$$\Delta = -\mathrm{tr}\,\nabla^2 = \nabla^* \nabla,$$

where ∇^* is the formal adjoint of ∇. The Lichnerowicz Laplacian acting on symmetric covariant two tensor is

$$\Delta_\mathrm{L} = \Delta + 2(\mathrm{Ric} - \mathrm{Sect}),$$

where

$$(\mathrm{Ric}\, u)_{ij} = \tfrac{1}{2}[\mathrm{Ric}(g)_{ik} u_j^k + \mathrm{Ric}(g)_{jk} u_i^k],$$

and

$$(\mathrm{Sect}\, u)_{ij} = \mathrm{Sect}(g)_{ikjl} u^{kl}.$$

As we will see, the essential spectrum is characterized near the boundary at infinity, it is thus convenient to define

$$M_\delta := \{x \in M, \rho(x) < \delta\},$$

which is a smooth manifold for δ small.

Throughout this article we work on an asymptotically hyperbolic manifold denoted by (M, g). The basic example of such a manifold is the hyperbolic space (B, g) where B is the open unit ball of $\mathbb{R}^n$ and $g = \rho^{-2}\bar{g}$ where $\bar{g}$ is the standard Euclidean metric and

$$\rho(x) = \tfrac{1}{2}(1 - |x|_{\bar{g}}^2).$$

L^2 denotes the usual Hilbert space of functions or tensors with the product (resp. norm)

$$\langle u, v\rangle_{L^2} = \int_M \langle u, v\rangle\, \mathrm{d}\mu_g \left(\mathrm{resp.} |u|_{L^2} = \left(\int_M |u|^2\, \mathrm{d}\mu_g\right)^{1/2}\right),$$

where $\langle u, v\rangle$ (resp. $|u|$) is the usual product (resp. norm) of functions or tensors relative to g, and the measure $\mathrm{d}\mu_g$ is the usual measure relative to g (we will omit the term $\mathrm{d}\mu_g$). H^2 denotes the usual Hilbert space of functions or tensors with two covariant derivatives in L^2 with the usual product and norm.

3. Semi-Fredholm criterion

Let P be an uniformly degenerate elliptic operator of order 2 on some tensor bundle over M (see [7] for more details). We give here a criterion for P to be semi-Fredholm. We first need the following definition.

Definition 3.1. We say that P satisfies the asymptotic estimate

$$\langle Pu, u\rangle_{L^2} \geq_\infty C|u|^2_{L^2} (\text{resp. } |Pu|_{L^2} \geq_\infty C|u|_{L^2})$$

if for all $\epsilon > 0$, there exist $\delta > 0$ such that, for all u smooth with compact support in M_δ, we have

$$\langle Pu, u\rangle_{L^2} \geq (C-\epsilon)|u|^2_{L^2} \ (\text{resp. } |Pu|_{L^2} \geq (C-\epsilon)|u|_{L^2}).$$

The proposition we will give now is standard in the context of non-compact manifolds, this estimate quantifies the basic quantum mechanical rule of thumb that the essential spectrum is determined solely by behavior at infinity.

Proposition 3.2. *Let $P: H^2 \to L^2$, P is semi-Fredholm* (*i.e. has finite dimensional kernel and closed range*) *if and only if P satisfies an asymptotic estimate*

$$|Pu|_{L^2} \geq_\infty c|u|_{L^2}$$

for some $c > 0$.

Proof. For the "if" part see [1, Proposition 2.7] for instance. For the "only if" part, I present here a proof given to me by J.M. Lee for completeness (I am grateful to him for allowing me to reproduce his argument). Suppose P is semi-Fredholm. Let Z be the orthogonal complement of Ker P and $Y := \operatorname{Im} P$. There exists a constant C such that

$$|u|_{H^2} \leq C|Pu|_{L^2} \quad \text{for all } u \in Z.$$

Let $u \in H^2$, $u = u_0 + u_Z$, where $u_0 \in \operatorname{Ker} P$ and $u_Z \in Z$. Let $\{v_1, \dots, v_m\}$ be an orthonormal basis of Ker P. Let δ be small and let χ be a smooth function equal to 1 on M_δ and supported in $M_{2\delta}$. We can choose δ small enough that $|\chi v_i|^2_{H^2} \leq (1/4m)|v_i|^2_{H^2} = 1/4m$ for $i = 1, \dots, m$. If supp $u \subset M_\delta$, we have

$$|u_0|^2_{H^2} = \sum_{i=1}^m \langle u, v_i\rangle^2_{H^2} = \sum_{i=1}^m \langle u, \chi v_i\rangle^2_{H^2} \leq \frac{1}{4}|u|^2_{H^2},$$

and therefore

$$|u|_{H^2} \leq |u_0|_{H^2} + |u_Z|_{H^2} \leq \tfrac{1}{2}|u|_{H^2} + |u_Z|_{H^2}.$$

Thus

$$|u|_{H^2} \leq 2|u_Z|_{H^2} \leq 2C|Pu|_{L^2},$$

which completes the proof. □

4. Asymptotic estimate

In this section, we recall a natural Laplacian acting on trace free symmetric covariant two tensors (see [8] or [3, 12.69, p. 355]), which will give a sharp estimate for the method used here. Note that the proof of Proposition 4.1 is inspired by Lee estimate on p-forms which can be found in [2, Lemma 6.3.3]. We now introduce some operators. Let us consider the operator from $\mathcal{S}_2$ to $\mathcal{T}_3$ defined by

$$(Du)_{kij} := \frac{1}{\sqrt{2}}(\nabla_k u_{ij} - \nabla_j u_{ik}).$$

This operator is nothing else than a constant times the covariant exterior differential (d^∇ see [3, 1.12., p. 24]) of u seen as a one-form with value in the cotangent bundle (tensors in the kernel of D are called Codazzi tensor [3, 16.3, p. 435]). The formal adjoint of D is

$$(D^*T)_{ij} = \frac{1}{2\sqrt{2}}(-\nabla^k T_{kij} - \nabla^k T_{kji} + \nabla^k T_{ijk} + \nabla^k T_{jik}).$$

We thus have

$$D^*Du_{ij} = -\nabla^k\nabla_k u_{ij} + \tfrac{1}{2}(\nabla^k\nabla_i u_{jk} + \nabla^k\nabla_j u_{ik}).$$

Now, consider the operator from $\mathcal{T}_1$ to $\mathcal{S}_2$

$$(L\omega)_{ij} := -\tfrac{1}{2}(\mathcal{L}_{w^\#}g)_{ij} = \tfrac{1}{2}(\nabla_i\omega_j + \nabla_j\omega_i),$$

and his formal adjoint

$$(L^*u)_i = (\operatorname{div} u)_i = -(\operatorname{Tr}\nabla u)_i = -\nabla^k u_{ki}.$$

We thus have

$$LL^*u_{ij} = -\tfrac{1}{2}(\nabla_i\nabla^k u_{jk} + \nabla_j\nabla^k u_{ik}).$$

A standard computation gives

$$\nabla^k\nabla_j u_{ik} - \nabla_j\nabla^k u_{ik} = \operatorname{Ric}(g)_{qj}u_i^q - \operatorname{Sect}(g)_{qilj}u^{ql}$$

and we obtain the Weitzenböck formula:

$$\Delta_0 := D^*D + LL^* = \nabla^*\nabla + \operatorname{Ric} - \operatorname{Sect}.$$

Proposition 4.1. *For $n > 2$, on $\mathcal{S}_{20}$, we have the asymptotic estimate*

$$\langle\Delta_0 u, u\rangle_{L^2} \geq_\infty \tfrac{1}{4}(n-3)^2|u|_{L^2}^2.$$

For the hyperbolic space, this estimate is global.

Proof. Let $u \in C_c^\infty(M\backslash\partial_0 M, \mathcal{S}_{20})$ and let v be a C^2 function on M, we will show that the inequality

$$\int_M |e^v D(e^{-v}u)|^2 + |e^{-v}L^*(e^v u)|^2 \geq 0 \tag{4.1}$$

gives the desired estimate for a “good” choice of v. We have

$$\mathrm{e}^{-v}L^*(\mathrm{e}^{v}u) = L^*u - \mathrm{d}v \times u, \quad \text{where } (\mathrm{d}v \times u)_i := (\nabla^k v)u_{ki}.$$

Besides, we have

$$\mathrm{e}^{v}D(\mathrm{e}^{-v}u) = Du - \mathrm{d}v \odot u, \quad \text{where } (\mathrm{d}v \odot u)_{kij} := \frac{1}{\sqrt{2}}[(\nabla_k v)u_{ij} - (\nabla_j v)u_{ik}],$$

so we obtain from Eq. (4.1)

$$\int_M |Du|^2 + |L^*u|^2 \underbrace{-2\langle Du,\ \mathrm{d}v \odot u\rangle - 2\langle L^*u,\ \mathrm{d}v \times u\rangle}_{\text{(I)}} + \underbrace{|\mathrm{d}v \times u|^2 + |\mathrm{d}v \odot u|^2}_{\text{(II)}} \geq 0. \tag{4.2}$$

(I) We have $\int_M \langle L^*u,\ \mathrm{d}v \times u\rangle = \int_M \langle u, L(\mathrm{d}v \times u)\rangle$, and

$$L(\mathrm{d}v \times u)_{ij} = \tfrac{1}{2}[(\nabla_j \nabla^k v)u_{ki} + (\nabla_i \nabla^k v)u_{kj} + \nabla^k v \nabla_j u_{ki} + \nabla^k v \nabla_i u_{kj}].$$

Moreover,

$$\begin{aligned}\langle Du,\ \mathrm{d}v \odot u\rangle &= \tfrac{1}{2}(\nabla_k u_{ij} - \nabla_j u_{ik})[(\nabla^k v)u^{ij} - (\nabla^j v)u^{ik}] \\ &= \nabla^k v(\nabla_k u_{ij})u^{ij} - \nabla^k v(\nabla_j u_{ik})u^{ij} \\ &= \nabla^k v(\nabla_k u_{ij})u^{ij} - \tfrac{1}{2}(\nabla^k v\nabla_j u_{ik} + \nabla^k v \nabla_i u_{jk})u^{ij},\end{aligned}$$

the last equality is due to the fact that u is symmetric. We obtain

$$\int_M \text{(I)} = \int_M -2\langle D_{\operatorname{grad} v}u, u\rangle - 2\langle H_v u, u\rangle,$$

where $(D_{\operatorname{grad} v})u_{ij} := \nabla^k v \nabla_k u_{ij}$ and $(H_v u)_{ij} := \frac{1}{2}[(\nabla_j \nabla^k v)u_{ki} + (\nabla_i \nabla^k v)u_{kj}]$.
Compute

$$\begin{aligned}A := \int_M \langle D_{\operatorname{grad} v}u, u\rangle &= \int_M \nabla^k v(\nabla_k u_{ij})u^{ij} \\ &= -\int_M u_{ij}[(\nabla_k \nabla^k v)u^{ij} + \nabla^k v \nabla_k u^{ij}] = \int_M \Delta v|u|^2 - A\end{aligned}$$

we thus have

$$2\int_M \langle D_{\operatorname{grad} v}u, u\rangle = \int_M \Delta v|u|^2.$$

Finally we get for the term (I):

$$\int_M \text{(I)} = \int_M -\Delta v|u|^2 - 2\langle H_v u, u\rangle.$$

(II) We compute

$$|\mathrm{d}v \odot u|^2 = \tfrac{1}{2}[(\nabla_k v)u_{ij} - (\nabla_j v)u_{ik}][(\nabla^k v)u^{ij} - (\nabla^j v)u^{ik}]$$
$$= |\mathrm{d}v|^2|u|^2 - |\mathrm{d}v \times u|^2.$$

For the term (II), we get

$$(\mathrm{II}) = |\mathrm{d}v|^2|u|^2.$$

Remark that $\Delta(\mathrm{e}^v) = \mathrm{e}^v(\Delta v - |\mathrm{d}v|^2)$ and the inequality (4.2) gives finally

$$\int_M \langle (D^*D + LL^*)u, u\rangle \geq \int_M \langle (\mathrm{e}^{-v}\Delta \mathrm{e}^v)u, u\rangle + 2\langle H_v u, u\rangle. \tag{4.3}$$

Take $v = s\ \ln\ \rho$, where $s \geq 0$, the Hessian of v is

$$H_{ij} = \nabla_i\nabla_j v = s(-\rho^{-2}\nabla_i\rho\nabla_j\rho + \rho^{-1}\nabla_i\nabla_j\rho).$$

Writing the difference between the Christoffel symbols of g and those of $\bar{g}$ is ($\bar{\nabla}$ denotes the Levi-Civita connexion relative to $\bar{g}$):

$$\Gamma_{ij}^k - \bar{\Gamma}_{ij}^k = \tfrac{1}{2}g^{ks}(\bar{\nabla}_i g_{sj} + \bar{\nabla}_j g_{is} - \bar{\nabla}_s g_{ij}) = -\rho^{-1}(\delta_i^k\bar{\nabla}_j\rho + \delta_j^k\bar{\nabla}_i\rho - \bar{g}_{ij}\bar{\nabla}^k\rho),$$

a straightforward calculation (recall that $|\mathrm{d}\rho|_{\bar{g}} = 1 + \mathrm{O}(\rho)$) gives

$$\nabla_i\nabla_j\rho = \rho^{-1}(2\nabla_i\rho\nabla_j\rho - \rho^2 g_{ij}) + \mathrm{O}(1).$$

We thus obtain

$$H_i^k = s(\rho^{-2}\nabla_i\rho\nabla^k\rho - \delta_i^k) + \mathrm{O}(\rho).$$

Hence

$$\langle H_v u, u\rangle = s(\rho^{-2}|\mathrm{d}\rho \times u|^2 - |u|^2) + \langle \mathrm{O}(\rho)u, u\rangle \geq [-s + \mathrm{O}(\rho)]|u|^2.$$

Furthermore we have $\mathrm{e}^{-v}\Delta\mathrm{e}^v = \rho^{-s}\Delta\rho^s = s(n-1-s) + \mathrm{O}(\rho)$, so from (4.3)

$$\int_M \langle (D^*D + LL^*)u, u\rangle \geq \int_M [s(n-3-s) + \mathrm{O}(\rho)]|u|^2,$$

where the parameter s is arbitrary, for $s = (n-3)/2$, we obtain the best estimate and conclude the proof for the general case.

For the hyperbolic space we can take a better function v. We first remark that the function $\varphi = \rho^{-1} - 1$ satisfies the Obata type equation:

$$\nabla_i\nabla_j\varphi = \varphi g_{ij}.$$

We thus take $v = -s\ \ln\ \varphi$, the Hessian of v is

$$H_{ij} = s(\varphi^{-2}\nabla_i\varphi\nabla_j\varphi - g_{ij}),$$

then

$$\langle H_v u, u\rangle = s(\varphi^{-2}|\mathrm{d}\varphi \times u|^2 - |u|^2) \geq -s|u|^2,$$

we also have

$$e^{-v}\Delta e^{v}=\varphi^{s}\Delta(\varphi^{-s})=-s\varphi^{-1}\Delta\varphi-s(s+1)\varphi^{-2}|d\varphi|_g^2$$
$$=-sn-s(s+1)\frac{1-2\rho}{(1-\rho)^2}\geq s(n-1-s),$$

when $s\geq 0$. Finally, from (4.3), we get for all $s\geq 0$,

$$\int_M\langle\Delta_0 u,u\rangle\geq s(n-3-s)\int_M|u|^2,$$

which completes the proof for the hyperbolic space when $s=(n-3)/2$. □

5. Non-existence of an asymptotic estimate

Proposition 5.1. *Let $\mathcal{K}$ be a zero order term which is an $O(\rho)$ near the boundary at infinity. Then for all $\lambda\geq(n-1)^2/4+2$ and for all $C>0$, the operator $P:=\Delta+\mathcal{K}-\lambda Id: H^2\to L^2$ cannot satisfy an asymptotic estimate*

$$|Pu|_{L^2}\geq_\infty C|u|_{L^2}.$$

Proof. Let $\lambda\geq(n-1)^2/4+2$, and $\mu:=\sqrt{\lambda-[(n-1)^2/4+2]}$. The idea of the proof is to construct a family of tensors $\{u_R\}$ with compact support in $M_{e^{-R/2}}$ such that $|Pu_R|_{L^2(M)}$ goes to zero when R goes to infinity but $|u_R|_{L^2(M)}$ goes to infinity when R goes to infinity.

It is well known (see [10, Lemma 5.1] for example) that we can change the defining function ρ into a defining function r such that the metric takes the form

$$g=r^{-2}\bar{g}=r^{-2}(dr^2+\hat{g}(r)),$$

on $M_\delta=]0,\delta[\times\partial_\infty M$ (reducing δ if necessary), where $\hat{g}(r)$ is a metric on $\{r\}\times\partial_\infty M$. Let $\hat{u}(0)$ be a smooth non-zero trace free symmetric covariant two tensor on $\partial_\infty M$, and $\bar{u}$ its parallel transported near $\partial_\infty M$ along geodesic $\bar{g}$-normal to $\partial_\infty M$. Reducing δ if necessary, we may assume that $\bar{u}$ is defined on M_δ. We have

$$\bar{u}=\hat{u}(r),$$

where $\hat{u}(r)$ is a trace free symmetric covariant two tensor on $\{r\}\times\partial_\infty M$.

Let us consider the functions $h(r):=r^{(n-5)/2}\cos(\mu\ \ln\ (r))$ and $f_R(r):=\Psi_R(r)h(r)$, where Ψ_R is as in Lemma A.1 and set

$$u_R:=f_R\bar{u}.$$

We first estimate the L^2-norm of u_R. For r small enough, we have

$$\tfrac{1}{2}|\hat{u}(0)|_{L^2(\partial_\infty M)}\leq|\hat{u}(r)|_{L^2(\{r\}\times\partial_\infty M)}\leq 2|\hat{u}(0)|_{L^2(\partial_\infty M)}.\tag{5.1}$$

Then, for R large enough,

$$\begin{aligned}|u_R|^2_{L^2} &= \int_M |u_R|^2_g \,\mathrm{d}\mu_g = \int_{\mathrm{e}^{-8R}}^{\mathrm{e}^{-R}} f_R^2 r^4 \left(\int_{\{r\}\times\partial_\infty M} |\hat{u}(r)|^2_{\hat{g}(r)} \,\mathrm{d}\mu_{\hat{g}(r)}\right) r^{-n}\,\mathrm{d}r \\ &\geq \frac{1}{4}|\hat{u}(0)|^2_{L^2(\partial_\infty M)} \int_{\mathrm{e}^{-8R}}^{\mathrm{e}^{-R}} \Psi_R^2 r^{-1} \cos^2(\mu \ln(r))\,\mathrm{d}r \\ &\geq \frac{1}{4}|\hat{u}(0)|^2_{L^2(\partial_\infty M)} \int_{\mathrm{e}^{-4R}}^{\mathrm{e}^{-2R}} r^{-1}\frac{\cos(2\mu \ln(r))+1}{2}\,\mathrm{d}r \\ &\geq \begin{cases} \frac{1}{2}|\hat{u}(0)|^2_{L^2(\partial_\infty M)} R & \text{if } \mu = 0 \\ \frac{1}{4}|\hat{u}(0)|^2_{L^2(\partial_\infty M)} \left[R + \frac{1}{4\mu}(\sin(8\mu R) - \sin(4\mu R))\right] & \text{if } \mu > 0 \end{cases}\end{aligned}$$

So we have

$$\lim_{R\to+\infty} |u_R|_{L^2} = +\infty.$$

Now, we estimate the L^2-norm of Pu_R. A straightforward calculation (see [7, Lemma 2.9, p. 202 and Proposition 2.7, p. 199] where here we have $(\bar{\nabla}^i r)\bar{u}_{ij} = 0$) gives

$$(\Delta + \mathcal{K})u_R = I(f_R)\bar{u} + rX(f_R),$$

where $I(f) := -r^2 f'' + (n-6)rf' + (2n-4)f$ and $X = \bar{a}r^2(\mathrm{d}^2/\mathrm{d}r^2) + \bar{b}r(\mathrm{d}/\mathrm{d}r) + \bar{c}$ is a second order operator polynomial in $r(\mathrm{d}/\mathrm{d}r)$ with $\bar{g}$-bounded coefficients depending on $\bar{g}$ and $\bar{u}$. Now as $I(h) = [(n-1)^2/4 + 2 + \mu^2]h$, we obtain

$$\begin{aligned}Pu_R &= [-r^2(\Psi_R'' h + 2\Psi_R' h') + (n-6)r\Psi_R' h]\bar{u} \\ &\quad + r[r^2(\Psi_R'' h + 2\Psi_R' h' + \Psi_R h'')\bar{a} + r(\Psi_R' h + \Psi_R h')\bar{b} + \Psi_R h\bar{c}].\end{aligned}$$

We will show that the L^2-norm of each term in the right part of the preceding equation goes to zero when R goes to infinity. For that, we use the fact that there exists some constant K_k such that $|r^k h^{(k)}| \leq K_k r^{(n-5)/2}$ for all $k \geq 0$ and $|r^k \Psi_R^{(k)}| \leq C_k/R$ for all $k \geq 1$ (see Lemma A.1). For the first three terms, we use the right part of inequality (5.1) too: the square of L^2-norm of the first term satisfies

$$\begin{aligned}\int_M r^4(\Psi_R'')^2 h^2 |\bar{u}|^2_g \,\mathrm{d}\mu_g &\leq \int_{\mathrm{e}^{-8R}}^{\mathrm{e}^{-R}} r^4 \frac{C_2^2}{r^4 R^2} r^{n-5} r^4 \left(\int_{\{r\}\times\partial_\infty M} |\hat{u}(r)|^2_{\hat{g}(r)}\,\mathrm{d}\mu_{\hat{g}(r)}\right) r^{-n}\,\mathrm{d}r \\ &\leq \frac{C_2^2}{R^2} 2|\hat{u}(0)|^2_{L^2(\partial_\infty M)} 7R.\end{aligned}$$

The same type of inequality for the two other terms shows their L^2-norm goes to zero when R goes to infinity. For the other terms, we remark that for r small enough,

$$\mathrm{vol}(\{r\}\times\partial_\infty M) := \int_{\{r\}\times\partial_\infty M} \mathrm{d}\mu_{\hat{g}(r)} \leq 2\,\mathrm{vol}(\partial_\infty M) := 2\int_{\partial_\infty M} \mathrm{d}\mu_{\hat{g}(0)}.$$

Thus, the square of L^2-norm of the 4th term satisfies

$$\int_M r^6 (\Psi_R'')^2 h^2 |\bar{a}|_g^2 \, \mathrm{d}\mu_g \leq \int_{\mathrm{e}^{-8R}}^{\mathrm{e}^{-R}} r^6 \frac{C_2^2}{r^4 R^2} r^{n-5} r^4 \sup_{\bar{M}} |\bar{a}|_{\bar{g}}^2 \left(\int_{\{r\}\times\partial_\infty M} \mathrm{d}\mu_{\hat{g}(r)} \right) r^{-n} \, \mathrm{d}r$$
$$\leq \frac{C_2^2}{R^2} \sup_{\bar{M}} |\bar{a}|_{\bar{g}}^2 2 \,\mathrm{vol}(\partial_\infty M) \frac{1}{2} (\mathrm{e}^{-2R} - \mathrm{e}^{-8R}).$$

The same type of inequality for the other terms shows that their L^2-norm goes to zero when R goes to infinity. From the triangle inequality, we can conclude:

$$\lim_{R\to+\infty} |Pu_R|_{L^2} = 0.$$

□

6. Conclusion

Theorem 6.1. *On an n-dimensional asymptotically hyperbolic manifold with $n > 2$, the essential spectrum of the Lichnerowicz Laplacian acting on trace free symmetric covariant two tensors is the ray*

$$[\tfrac{1}{4}(n-1)(n-9), +\infty[.$$

For the hyperbolic space, this is the spectrum.

Proof. Recall that the Lichnerowicz Laplacian is

$$\Delta_{\mathrm{L}} = \Delta + 2(\mathrm{Ric} - \mathrm{Sect}) = \Delta_0 + \mathrm{Ric} - \mathrm{Sect},$$

and that (see [12] for instance)

$$\mathrm{Sect}(g)_{ijkl} = g_{ik}g_{jl} - g_{il}g_{jk} + \mathrm{O}(\rho^{-3}).$$

We thus have $\mathrm{Ric} - \mathrm{Sect} = -nId + \mathcal{K}$, where $\mathcal{K} = \mathrm{O}(\rho)$ for the general case and $\mathcal{K} = 0$ for the hyperbolic space case.

From Proposition 4.1, for all $\lambda < (n-1)(n-9)/4 = (n-3)^2/4 - n$, we obtain

$$|u|_{L^2} |(\Delta_L - \lambda Id)u|_{L^2} \geq \langle (\Delta_L - \lambda Id)u, u\rangle_{L^2} \geq_\infty [\tfrac{1}{4}(n-3)^2 - n - \lambda)|u|_{L^2}^2,$$

so from Proposition 3.2 $\Delta_{\mathrm{L}} - \lambda Id$ is semi-Fredholm.

From Propositions 3.2 and 5.1, for all $\lambda \geq (n-1)(n-9)/4 = (n-1)^2/4 + 2 - 2n$, $\Delta_{\mathrm{L}} - \lambda Id = \Delta + 2\mathcal{K} - (\lambda + 2n)Id$ is not semi-Fredholm.

Recall that the essential spectrum of Δ_{L} is the closed set

$$\sigma_{\mathrm{e}}(\Delta_L) = \{\lambda \in \mathbb{R}, \quad \Delta_L - \lambda Id \,\text{is not semi-Fredholm}\},$$

so the theorem follows for the general case.

For the hyperbolic space, the global estimate of Proposition 3.2 shows there cannot exist some eigenvalues smaller than $(n-1)(n-9)/4$ and the spectrum is $\sigma(\Delta_{\mathrm{L}}) = \sigma_{\mathrm{e}}(\Delta_{\mathrm{L}})$. □

Remark 6.2. The same argument gives (on trace free symmetric two tensor):

$$\sigma_{\mathrm{e}}(\Delta) = [\tfrac{1}{4}(n-1)^2 + 2, +\infty[, \qquad \sigma_{\mathrm{e}}(\Delta_0) = [\tfrac{1}{4}(n-3)^2, +\infty[,$$

and in each case, $\sigma = \sigma_{\mathrm{e}}$ for the hyperbolic space.

Acknowledgements

I am grateful to L. Andersson and J.M. Lee for useful conversations, to P.T. Chruściel and Ph. Delanoë for their comments on the original manuscript and to the Royal Institute of Technology of Stockholm for its hospitality at the beginning of this work.

Appendix A

In this appendix, we recall a nice family of cutoff functions which can be found in [1, Definition 21, p. 1362].

Lemma A.1. *Let* (M, g, ρ) *be an asymptotically hyperbolic manifold. For* $R \in \mathbb{R}$ *large enough, there exits a cutoff function* $\Psi_R : M \to [0, 1]$ *depending only on* ρ, *supported in the annulus* $\{\mathrm{e}^{-8R} < \rho < \mathrm{e}^{-R}\}$, *equal to* 1 *in* $\{\mathrm{e}^{-4R} < \rho < \mathrm{e}^{-2R}\}$ *and which satisfies for* R *large*:

$$\left|\frac{\mathrm{d}^k \Psi_R}{\mathrm{d}\rho^k}(\rho)\right| \le \frac{C_k}{R\rho^k},$$

for all $k \in \mathbb{N}\backslash\{0\}$, *where* C_k *is independent of* R.

Proof. Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth function equal to 1 on $]-\infty, 1]$ and 0 on $[2, +\infty[$. We define

$$\chi_R(x) := \chi\left(\frac{\ln(\rho(x))}{-R}\right),$$

we then have $\chi_R : M \to [0, 1]$ is equal to 1 on $\rho \ge \mathrm{e}^{-R}$ and 0 on $\rho \le \mathrm{e}^{-2R}$. Now we define

$$\Psi_R := \chi_{4R}(1 - \chi_R)$$

which satisfies the desired properties. □

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